

Nilpotent classical mechanics: s-geometry.

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Abstract

We introduce specific type of hyperbolic spaces. It is not a general linear covariant object, but of use in constructing nilpotent systems. In the present work necessary definitions and relevant properties of configuration and phase spaces are indicated. As a working example we use a D=2 isotropic harmonic oscillator.

1 Introduction

Typical configuration space of a classical mechanical system is endowed with the Riemannian structure if not more commonly and simply with a flat Euclidean one. Here our aim is to study the case of specific subclass of pseudo-Riemannian configuration spaces, having zero trace metric. We want to restrict ourselves here, even further, to the case of the strictly-traceless forms which we shall call the s-forms. Definition of these notions will be given in the next section. Motivation for this objects has roots in the nilpotent mechanical systems cf. Ref. [1]. In the present work we look how such a s-form defined structure influences the behaviour of mechanical system. Because lowest possible dimension for s-geometry is equal two, we consider the

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D=2 isotropic oscillator (D=2 IHO) and compare it to the conventional, Riemannian picture. This gives interesting interpretation of the set of integrals of motion. For all types of geometries defined by non-degenerate symmetric forms the equations of motion are the same, and the set of integrals of motion in the phase space is preserved, but the role of particular integrals changes. In respective pictures - we switch between $su(2)$ and $su(1, 1)$ Lie algebras.

2 s-forms

Let \mathbb{V} be a vector space $\dim \mathbb{V} = 2n$, over \mathbb{R} . By s -form we shall understand a \mathbb{R} -bilinear, symmetric mapping s

$$s : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$$

which is weakly nondegenerate (i.e. $s(v, v') = 0$ for all $v' \in \mathbb{V}$, then $v = 0$) and strictly traceless in the following sense: there exists a basis $\{b_i\}_1^{2n}$ of \mathbb{V} such that

$$s(b_i, b_i) = 0, \forall i \quad (1)$$

Let us note here that for antisymmetric form we have automatically that it is strictly traceless, moreover it is strictly traceless for any sets of vectors. The standard s -form is defined by

$$s(v, v') = \sum_{i=1}^n v_i v'_{n+i} + v'_i v_{n+i} \quad (2)$$

We shall call a basis $\{e_i\}_1^{2n}$ the s -admissible basis iff $s(e_i, e_i) = 0, \forall i$. Now we are ready to define the s -space. By the s -space we will understand a linear space \mathbb{V}_{2n} with a s -form and the set of all s -admissible bases i.e. the triple (\mathbb{V}_{2n}, s, B) . From this definition it is clearly seen that s -space is not general linear covariant and its symmetry group is essentially narrower.

Two s -spaces (\mathbb{V}, s, B) and (\mathbb{V}', s', B') are isomorphic if there exists a vector space isomorphism Φ such that

$$s'(\phi(v), \phi(w)) = s(v, w), \quad \text{where } v, w \in \mathbb{V} \quad (3)$$

In particular we have that $\phi(B) = B'$. Let $Aut_s(\mathbb{V})$ be the set of automorphisms of (\mathbb{V}, s, B) . For the s -spaces of the same dimension respective automorphism groups are isomorphic.

2.1 s -orthogonality

The s -form is not positive definite and in fact is related to the hyperbolic type of geometry. One can think about "relative length" of vectors, in the following sense: $s(b_i, b_i)$ vanishes but we can normalize non-vanishing product of vectors from a fixed set, e.g. $s(b_i, b_{i+n}) = 1$, $i = 1, 2, \dots, n$. We can use the notion of orthogonality, as usual, and structure of orthogonal subspaces is somehow similar to the case of the symplectic geometry [2], but different. We will introduce relevant subspaces in analogy to the symplectic geometry. Let (\mathbb{V}, s, B) be a s -space and $W \subset \mathbb{V}$

$$W^\perp = \{w \in \mathbb{V} \mid s(v, w) = 0, \forall w \in W\} \quad (4)$$

As always if $W_1 \subset W_2$ then $W_2^\perp \subset W_1^\perp$ and $(W^\perp)^\perp \subset W$. Using orthogonality, in analogy to the symplectic geometry [2], for the s -spaces we can distinguish the following types of subspaces:

- (1) s -isotropic, if $W \subset W^\perp$
- (2) s -coisotropic, if $W^\perp \subset W$
- (3) s -Lagrangian, if $W = W^\perp$

Example:

$n = 1$, (\mathbb{R}^2, per, B) , $\{b_i\}_1^2 \in B$, where per is a permanent of the matrix $\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$. Then $\text{Span}\{b_1\}$ and $\text{Span}\{b_2\}$ are s -Lagrangian subspaces. Let us note that in analogous example of the symplectic space (\mathbb{R}^2, det) any 1-dimensional subspace is Lagrangian.

3 s -plectic group $Ap(n)$

It is natural to ask what is the symmetry group of this sub-type of hyperbolic geometry. The condition that trace of the s -form is strictly 0 (i.e. $Tr(s)|_k = 0$, $k = 1, \dots, 2n$) restrict essentially the group of transformations.

Let $\mathcal{S} = (\mathbb{R}^{2n}, s, B)$ be a standard s -space. The group of all automorphisms of \mathcal{S} preserving the s -form (2) will be denoted $Ap(n)$ and called the standard s -plectic group. Naturally the $Ap(n)$ is isomorphic to s -plectic group $Ap(\mathbb{V}, s)$

for any $2n$ -dimensional s -space. For further convenience we will work with matrices. Let form s be represented by the following matrix

$$s = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}; \quad s^2 = \mathbb{I}_{2n}; \quad s^T = s; \quad Tr(s)|_k = 0, \quad k = 1, 2, \dots, 2n \quad (5)$$

The \mathbb{I}_n denotes n -dimensional unit matrix and $Tr(s)|_k$ the trace of a principal $k \times k$ block of the matrix s . Now, $D \in Ap(n)$ when

$$D^T s D = s. \quad (6)$$

In a slightly restricted way we can work in analogy to the symplectic case. Let s be given in natural basis, Eq.(5), then we can write $D \in Ap(n)$ in a block form

$$D = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad (7)$$

with P, Q, R, S being $n \times n$ -blocks. The condition (6) enforces the following relations

$$R^T P = -P^T R, \quad R^T Q + P^T S = \mathbb{I}_n \quad (8)$$

$$S^T Q = -Q^T S, \quad S^T P + Q^T R = \mathbb{I}_n \quad (9)$$

and moreover $(\det D)^2 = 1$ (unlike the symplectic case here the sign of $\det D$ depends on dimension of s -space).

Example:

Let $D \in Ap(1)$,

$$D = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad (10)$$

The conditions (8) and (9) in this case have two solutions

$$A = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}, \quad p, q \in \mathbb{R} \quad (11)$$

These matrices generate $Ap(1)$, a subgroup is generated by B . As it is easily seen: $\det A = -1$, $\det B = 1$.

4 Classical mechanics in s-spaces

As a first application of s -spaces let us consider the formalism of classical mechanics, where original configuration space is not Euclidean or Riemannian, but is equipped with the flat s -form, so the lowest possible dimension is equal two and symmetry group is restricted to translations and $Ap(n)$ transformations. Since the form s is symmetric and non-degenerate, the formulation of conventional tools of the classical mechanics mostly remains valid. As a fundamental system for this type of mechanics, modeled on \mathbb{R}^{2n} we shall consider the two dimensional harmonic oscillator, for simplicity - isotropic.

4.1 Configuration space

To define a mechanical system we shall introduce a Lagrangian with terms being the analogs of kinetic and potential energy. Analogs only, because the s -form is not positive definite.

$$L = \frac{m}{2}s(\dot{x}, \dot{x}) - V(x) \quad (12)$$

In scope of the further example of the IHO, let

$$V(x) = \frac{m\omega}{2}s(x, x) = \frac{m\omega}{2}s_{ij}x^i x^j. \quad (13)$$

The form of the Euler-Lagrange equations is conventional, but due to the new geometry, an interpretation of motions of the system gets different.

4.2 Phase space

Passage to the phase space via the Legendre transformation is straightforward and we get Hamiltonian containing s -form

$$H = \frac{1}{2m}(s^{-1})^{ij}p_i p_j + \frac{m\omega^2}{2}s_{ij}x^i x^j \quad (14)$$

and conventional equations of motion

$$\begin{cases} \dot{x}^i &= \frac{1}{m}(s^{-1})^{ij}p_j \\ \dot{p}_i &= -m\omega^2 s_{ij}x^j \end{cases} \quad (15)$$

The form of the Poisson brackets is standard, where contraction between momenta and coordinates is, as usual, done by means of natural pairing in

cotangent bundle over the configuration space. Using the s -form we can dualize coordinates, to have all phase space indices on one level $x_i = s_{ij}x^j$. Then natural symplectic form Ω on such a phase space will be given in symplectic basis by the following matrix

$$\Omega = \begin{pmatrix} 0 & s^{ij} \\ -s^{ij} & 0 \end{pmatrix} \quad (16)$$

What is obviously true, for any non-degenerate symmetric form, like Euclidean δ , pseudo-Euclidean η , as well as for the strictly traceless form s .

4.3 D=2 IHO

Using symplectic coordinates (x_i, p_i) , we have the following Hamiltonian for isotropic harmonic oscillator in $D = 2$

$$H = \frac{1}{2m}(s^{-1})^{ij}p_i p_j + \frac{m\omega^2}{2}(s^{-1})^{ij}x_i x_j \quad (17)$$

and Hamilton equations of motion, whose form does not depend on the used symmetric form: $s_{ij}, \delta_{ij}, g_{ij}$

$$\begin{cases} \dot{x}_i &= \frac{1}{m}p_i \\ \dot{p}_i &= -m\omega^2 x_i \end{cases} \quad (18)$$

Let us consider the following phase space functions [3, 4]

$$H_0 = \frac{1}{2m}(p_1^2 + p_2^2) + m\omega^2(x_1^2 + x_2^2) \quad (19)$$

$$H_1 = \frac{1}{m}(p_1 p_2) + m\omega^2 x_1 x_2 \quad (20)$$

$$H_2 = \frac{1}{2m}(p_2^2 - p_1^2) + m\omega^2(x_2^2 - x_1^2) \quad (21)$$

$$H_3 = \omega(x_1 p_2 - x_2 p_1) \quad (22)$$

They are integrals of motion not only for the Euclidean $D = 2$ oscillator, but also for other geometries like hyperbolic, including s -geometry. This property survives also generalization to the curved case, to any two-dimensional spaces of constant curvature - the curved harmonic oscillator is superintegrable in

all such cases [5]. Above functions are related to the components of Jauch-Hill-Fradkin tensor ([6, 7]) and saturate the following identity

$$H_0^2 - H_1^2 - H_2^2 - H_3^2 = 0 \quad (23)$$

In particular we can isolate three cases which are realized by this set of phase-space integrals of motions, related to different geometries of the original configuration space:

- Euclidean - with the Hamiltonian H_0 . The integrals of motion (H_1, H_2, H_3) generate the algebra isomorphic to the $su(2)$ and the standard symplectic form

$$\Omega_e = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \quad \text{where} \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (24)$$

- Hyperbolic - with the Hamiltonian H_2 . The integrals of motion (H_0, H_1, H_3) generate the algebra isomorphic to the $su(1, 1)$ and the symplectic form in this case is the following

$$\Omega_h = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \quad \text{where} \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (25)$$

- s-Hyperbolic - with the Hamiltonian H_1 . The integrals of motion (H_0, H_2, H_3) generate the algebra isomorphic to the $su(1, 1)$ and the symplectic form in this case is the following

$$\Omega_s = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}, \quad \text{where} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (26)$$

The symplectic forms are given in bases analogous to that used for Ω given by Eq.(16), where δ , η , s are metrics defining respective geometries of the configuration spaces.

Conclusions

An interesting observation is that phase space description of the system is not that sensitive to the specific geometry of the configuration space, in particular question of integrability of the system can be answered in unified way including various types of geometry [5]. In the forthcoming paper we will use the introduced s -geometry to describe essentially nilpotent mechanical systems.

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